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DISJOINT SHORTEST PATHS IN GRAPHS

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It is an interesting problem that how much connectivity ensures the existence of n disjoint paths joining given n pairs of vertices, but to get a sharp bound seems to be very difficult. In this paper, we study how much *geodetic connectivity* ensures the existence of n disjoint *geodesics* joining given n pairs of vertices, where a graph is called k-geodetically connected if the removal of any $k-1$ vertices does not change the distance between any remaining vertices.

In this paper we consider finite undirected graphs without loops and multiple edges. We say that a graph G is n-linked if G has at least $2n$ vertices and for any $2n$ distinct vertices $x_1, \ldots, x_n, y_1, \ldots, y_n$ there exist n disjoint paths $P_1, \ldots, P_n$ such that P_i joins x_i and y_i $(1 \leqq i \leqq n)$. It is interesting how much connectivity ensures n-linkedness. Larman and Mani [4] and Jung [3] showed that 2^{3n}-connected graphs are n-linked, that is, the following function is well-defined: $f(n)=\min\{k|\text{every } k\text{-connected graph is } n\text{-linked}\}$. Thomassen [5] showed that $f(2)=6$ and conjectured that $f(n)==2n+2$ for $n \geqq 2$. At present, no polynomial function is known that is an upper bound of f. In this paper we study how much geodetic-connectivity, which is first introduced by Entringer, Jackson and Slater [2], is required for graphs to have n disjoint shortest paths joining given pairs of vertices.

Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G, respectively. We denote by $P=[x=a_0, a_1, \ldots, a_l=y]$ the path which goes through vertices $a_0, a_1, \ldots, a_l$. We also write $P=P[x; y]$. The number l is called the length of P and is denoted by $l(P)$. Let x and y be vertices of a graph G. We denote by $d_G(x, y)$ the distance between x and y in G and by $\Gamma_G(x)$ the set of vertices adjacent to x in G. Let P_1 and P_2 be paths in a graph G. We say that P_1 and P_2 are *disjoint* if $V(P_1) \cap V(P_2)=\emptyset$. Let $\{P_1, \ldots, P_n\}$ be a set of paths in G. Then $\{P_1, \ldots, P_n\}$ is called a *linkage* if any two distinct members P_i and P_j are disjoint. If u and v are (not necessarily distinct) vertices of a graph G, a shortest path joining u and v is called a *uv-geodesic.* A linkage $\{P_1, \ldots, P_n\}$ is called a *geodetic linkage* if each P_i is a geodesic. Notation not defined here is found in [1]. Let n be a nonnegative integer. A graph G is *n-geodetically connected* if $d_{G-S}(x, y)=d_G(x, y)<\infty$ for any $S \subset V(G)$ satisfying $|S|=n-1$ and for any $x, y \in V(G-S)$, where $G-S$ is the

AMS subject classification (1980): 05 C 40; 05 C 38 subgraph of G obtained from G by deleting S. Using this notation, our main result is stated as follows:

Theorem 1. *Let $n\geqq 1$ and $x_1, \ldots, x_n, y_1, \ldots, y_n$ be distinct vertices of G. Suppose $|\{i|d_G(x_i, y_i)=2\}|=q\geqq 1$ and G is $(2n+q-2)$-geodetically connected. Then there exists a geodetic linkage $\{P_1[x_1; y_1], \ldots, P_n[x_n; y_n]\}$.*

In order to prove Theorem 1, we use a characterization of n-geodetically connected graphs by Entringer, Jackson and Slater [2].

Theorem A [2]. *Let G be a graph. Then the following conditions are equivalent.*

(1) *G is n-geodetically connected.*

(2) *If $x, y\in V(G)$ and $d_G(x, y)=2$, then $|\Gamma_G(x)\cap\Gamma_G(y)|\geqq n$.* ▮

The following two lemmas are frequently used in the proof of Theorem 1.

Lemma 1. *Let $x, y, z\in V(G)$ and $P=P[x; y]$ be an xy-geodesic. Then $|\Gamma_G(z)\cap \cap V(P)|\leqq 3$.*

Proof. Let $P=[x=a_0, a_1, \ldots, a_l=y]$. Suppose $|\Gamma_G(z)\cap V(P)|\geqq 4$. Then there exist two vertices $a_i, a_j\in V(P)\cap\Gamma_G(z)$ satisfying $d_G(a_i, a_j)>2$. Let $P'=[x=a_0, a_1, \ldots, \ldots, a_i, z, a_j, \ldots, a_l=y]$. Then P' is shorter than P. This is a contradiction. ▮

By a similar argument, we obtain the following lemma.

Lemma 2. *Let $x, y, a, b\in V(G)$ and $P=P[x; y]$ be an xy-geodesic in G. If a and b are adjacent, $|(\Gamma_G(a)\cup\Gamma_G(b))\cap V(P)|\leqq 4$.* ▮

Now we prove Theorem 1.

Proof of Theorem 1. The proof is done by induction on n. It is obvious that the theorem holds when $n=1$. Suppose $n\geqq 2$. Since $q\geqq 1$, we may assume $d_G(x_1, y_1)= =2$. Then there exists a geodetic linkage $\{R_1[x_1; y_1], \ldots, R_{n-1}[x_{n-1}; y_{n-1}]\}$ in $G-\{x_n, y_n\}$ by the induction hypothesis. Since $2n+q-2\geqq 3$, $d_{G-\{x_n, y_n\}}(x_i, y_i)= =d_G(x_i, y_i)$ and each R_i is a geodesic in G. If $d_G(x_n, y_n)=1$, the theorem follows easily. Hence we may assume $d_G(x_n, y_n)\geqq 2$. Let Φ be the family of ordered sets $(z_1, z_2, Q_1, Q_2, P_1, \ldots, P_{n-1})$, where $z_1, z_2\in V(G)$, $Q_1=Q_1[x_n; z_1]$ and $Q_2= =Q_2[y_n; z_2]$ are geodesics and each P_i is an $x_i y_i$ geodesic $(1\leqq i\leqq n-1)$, satisfying the following conditions.

(1) The set $\{Q_1, Q_2, P_1, \ldots, P_{n-1}\}$ is a geodetic linkage.

(2) $$d_G(x_n, z_1)+d_G(z_1, z_2)+d_G(z_2, y_n) = d_G(x_n, y_n).$$

Let $Q_1^{(0)}$ and and $Q_2^{(0)}$ be the trivial $x_n x_n$- and $y_n y_n$-geodesics. Then $(x_n, y_n, Q_1^{(0)}, Q_2^{(0)}, R_1, \ldots, R_{n-1})\in\Phi$. So $\Phi\neq\emptyset$. Take $(z_1, z_2, Q_1, Q_2, P_1, \ldots, P_{n-1})\in\Phi$ such that $d_G(z_1, z_2)$ is minimum in Φ. Let $r_1=d_G(x_n, z_1)$, $r_2=d_G(y_n, z_2)$, $s=d_G(z_1, z_2)$ and $d_i=d_G(x_i, y_i)$ $(1\leqq i\leqq n)$. Let $Q_1=[x_n=c_0, c_1, \ldots, c_{r_1}=z_1]$. If $s=0$, then $z_1=z_2$ and the theorem follows. The case $s=1$ does not occur. Hence we may assume $s\geqq 2$. Then there exists $z\in V(G)$ such that $d_G(z_1, z)=2$ and $d_G(z, z_2)=s-2$. Let $U=\Gamma_G(z_1)\cap\Gamma_G(z)=\{u_1, \ldots, u_h\}$. By the assumption, $h\geqq 2n+q-2$. If $U\not\subset\bigcup_{i=1}^{n-1} V(P_i)$, say $u_k \in U - \bigcup_{i-1}^{n-1} V(P_i)$, set $Q' = [x_n = c_0, \ldots, c_{r_1}, u_k]$. Then $(u_k, z_2, Q', Q_2, P_1, \ldots, \ldots, P_{n-1}) \in \Phi$ and $d_G(u_k, z_2) = s-1$. This contradicts the assumption. Hence $U \subset \subset \bigcup_{i=1}^{n-1} V(P_i)$. On the other hand, $|U \cap V(P_i)| \leqq 3$ for any i, $1 \leqq i \leqq n-1$, by Lemma 1. Let $L = |\{i \mid 1 \leqq i \leqq n-1, |U \cap V(P_i) = 3\}|$. Then $3L + 2(n-1-L) \geqq h \geqq 2n+q-2$, which implies $L \geqq q$. In particular, $L \geqq 1$.

We claim that $d_j = 2$ if $|U \cap V(P_j)| = 3$. Assume $|U \cap V(P_j)| = 3$ and $d_j \geqq 3$. Let $P_j = [x_j = b_0, \ldots, b_{d_j} = y_j]$. We may assume that $b_{l-1} = u_1$, $b_l = u_2$ and $b_{l+1} = u_3$ for some l. By the assumption $l \geqq 2$ or $l \leqq d_j - 2$. Without loss of generality we may assume $l \geqq 2$. Set $W_1 = \Gamma_G(b_{l-1}) \cap \Gamma_G(b_{l+1})$ and $W_2 = \Gamma_G(b_{l-2}) \cap \Gamma_G(b_l)$. Note that $W_1 \cap W_2 \neq \emptyset$ since $d_G(x_j, w_1) = l$ for any $w_1 \in W_1$ and $d_G(x_j, w_2) = l-1$ for any $w_2 \in W_2$. Since $b_{l-1}, b_l, b_{l+1} \in U$, $d_G(x_n, b_i) = r_1 + 1$ $(i = l-1, l, l+1)$. Hence $V(Q_1) \cap W_1 = \{z_1\}$. Similarly, $V(Q_2) \cap W_1 \subset \{z_2\}$. Suppose there exists a vertex $w \in W_1$ such that $w \notin \bigcup_{l=1}^{n-1} V(P_i) \cup \{z_1, z_2\}$. Then taking

$$P_j' = [x_j = b_0, \ldots, b_{l-1}, w, b_{l+1}, \ldots, b_{d_j} + y_j],$$

$$P_i' = P_i (i \neq j)$$

and

$$Q' = [x_n = c_0, \ldots, c_{r_1}, u_2 = b_l],$$

we have $(u_2, z_2, Q', Q_2, P_1', \ldots, P_{n-1}') \in \Phi$ and $d_G(u_2, z_2) = s-1$. This contradicts the assumption. Hence $W_1 \subset \bigcup_{i=1}^{n-1} V(P_i) \cup \{z_1, z_2\}$. Similarly $W_2 \subset \bigcup_{i=1}^{n-1} V(P_i) \cup \{z_2\}$. Therefore, we have $W_1 \cup W_2 \subset \bigcup_{i=1}^{n-1} V(P_i) \cup \{z_1, z_2\}$. Clearly $|W_1 \cap V(P_j)| = = |W_2 \cap V(P_j)| = 1$. Since $W_1 \subset \Gamma_G(b_{l+1})$ and $W_2 \subset \Gamma_G(b_l)$, $|(W_1 \cup W_2) \cap V(P_i)| \leqq \leqq |(\Gamma_G(b_{l+1}) \cup \Gamma_G(b_l)) \cap V(P_i)| \leqq 4$ for any i, $1 \leqq i \leqq n-1$, by Lemma 2. Thus we have

$$|W_1| + |W_2| \leqq 3q + 4\{(n-2) - q\} + 1 + 1 + 2 = 4n - q - 4.$$

On the other hand, since $W_1 \cap W_2 = \emptyset$, $|W_1| + |W_2| \geqq 2(2n+q-2)$. Then we have $q \leqq 0$. This contradicts the assumption. Hence $d_j = 2$ if $|U \cap V(P_j)| = 3$. Thus $L \leqq q$ and we have $L = q$. Therefore, $d_n \geqq 3$.

Next, we take a vertex z' such that $d_G(z_2, z') = 2$ and $d_G(z_1, z') = s-2$. Let $U' = \Gamma_G(z_2) \cap \Gamma_G(z')$ and $L' = |\{i \mid 1 \leqq i \leqq n-1, |U' \cap V(P_j)| = 3\}|$. Then by the similar argument as above, we have $L' = q$. This implies $s = 2$.

Since $L = q$ and $d_G(x_1, y_1) = 2$, we have $V(P_1) \subset \Gamma_G(z_1) \cap \Gamma_G(z_2)$. Furthermore, $r_1 \geqq 1$ or $r_2 \geqq 1$ since $d_n \geqq 3$. We may assume $r_1 \geqq 1$. Let $P_1 = [u_{i_1}, u_{i_2}, u_{i_3}]$, where $u_{i_1} = x_1$ and $u_{i_3} = y_1$. Let $U' = \Gamma_G(c_{r_1 - 1}) \cap \Gamma_G(u_{i_2})$. Since $d_G(c_{r_1-1}, u_{i_2}) = 2$, $|U'| \geqq 2n+q-2$. Note that $U \cap U' = \emptyset$. If $u \notin \bigcup_{i-1}^{n-1} V(P_i) \cup \{z_1\}$ for some $u \in U'$, then take

$$P_1' = [u_{i_1}, z_1, u_{i_3}],$$

$$P_i' = P_i (2 \leqq i \leqq n-1)$$

and

$$Q' = [x_n = c_0, \ldots, c_{r-1}, u, u_{i_2}].$$

Then $(u_{i_2}, z_2, Q', Q_2, P'_1, \ldots, P'_{n-1}) \in \Phi$ and $d_G(u_{i_2}, z_2) = s-1$. This contradicts the assumption. Therefore $U' \subset \bigcup_{i=1}^{n-1} V(P_i) \cup \{z_1\}$ and we have $U \cup U' \subset \bigcup_{i=1}^{n-1} V(P_i) \cup \cup \{z_1\}$. Since $U \cap U' = \emptyset$ and $|(U \cup U') \cap V(P_i)| \leqq |(\Gamma_G(z_1) \cup \Gamma_G(u_{i_2})) \cap V(P_i)| \leqq 4$ $(1 \leqq i \leqq n-1)$, $3q + 4(n-q-1) + 1 \geqq |U| + |U'| \geqq 2(2n+q-2)$, which implies $3q \leqq 1$. This contradicts the assumption. Hence the result follows. ∎

When $q=0$, one more geodetic-connectivity is needed.

Theorem 2. *Let $n \geqq 1$ and $x_1, \ldots, x_n, y_1, \ldots, y_n$ be distinct vertices of a graph G satisfying $d_G(x_i, y_i) \neq 2$ $(1 \leqq i \leqq n)$. If G is $(2n-1)$-geodetically connected, there exists a geodetic linkage $\{P_1[x_1; y_1], \ldots, P_n[x_n; y_n]\}$.* ∎

The proof of Theorem 2 is similar to that of Theorem 1 and we omit it.

Let $n \geqq 1$. A graph G is called *geodetically n-linked* if $|V(G)| \geqq 2n$ and for any distinct $2n$ vertices $x_1, \ldots, x_n, y_1, \ldots, y_n$ of G there exists a geodetic linkage $\{P_1[x_1; y_1], \ldots, P_n[x_n; y_n]\}$. As a corollary of Theorem 1 and Theorem 2 we obtain the following.

Corollary. *Every $(3n-2)$-geodetically connected graph is geodetically n-linked.*

Proof. Suppose G is $(3n-2)$-geodetically connected. Let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be distinct vertices of G. Set $q = |\{i \mid 1 \leqq i \leqq n, d_G(x_i, y_i) = 2\}|$. Since $n \geqq 1$ and $q \leqq n$, $3n-2 \geqq \max\{2n+q-2, 2n-1\}$. Hence the result follows by Theorems 1 and 2. ∎

Now we prove that the results of Theorem 1 and Theorem 2 are best possible.

Theorem 3. *Let n and q be positive integers such that $q \leqq n$. Then there exist a graph $G(n, q)$ and $2n$ vertices $x_1, \ldots, x_n, y_1, \ldots, y_n$ of $G(n, q)$ satisfying the following conditions.*

(i) $d_G(x_i, y_i) = 2$ *for any is such that* $1 \leqq i \leqq q$.

(ii) $G(n, q)$ *is* $(2n+q-3)$*-geodetically connected.*

(iii) *There exists no geodetic linkage* $\{P_1[x_1; y_1], \ldots, P_n[x_n; y_n]\}$.

Proof. Let K be a complete graph of order $2n+q-1$ whose vertex set is $\{x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_{q-1}\}$. Define $G(n, q)$ by $G(n, q) = K - \{x_1y_1, \ldots, x_qy_q\}$. In $G(n, q)$, $\{x_1, y_1\}, \ldots, \{x_q, y_q\}$ are the only pairs of vertices of distance two and $\Gamma_G(x_i) = \Gamma_G(y_i) = V(G(n, q)) - \{x_i, y_i\}$ for $1 \leqq i \leqq q$. Hence $G(n, q)$ is $(2n+q-3)$-geodetically connected by Theorem A. But there does not exists a desired geodetic linkage since each geodesic joining x_i and y_i $(1 \leqq i \leqq q)$ must pass through some z_i. ∎

Theorem 4. *Let n be a positive integer. Then there exist a graph $G(n)$ and $2n$ vertices $x_1, \ldots, x_n, y_1, \ldots, y_n$ of $G(n)$ satisfying the following conditions*

(i) $d_{G(n)}(x_i, y_i) = 3$ $(1 \leqq i \leqq n)$.

(ii) *$G(n)$ is $(2n-2)$-geodetically connected.*

(iii) *There exists no geodetic linkage $\{P_1[x_1; y_1], \ldots, P_n[x_n; y_n]\}$.*

Proof. We define $V(G(n))$ and $E(G(n))$ as follows.

$$V(G(n)) = \{x_1, \ldots, x_n\} \cup \{a_1, \ldots, a_{n-1}\} \cup \{y_{jk}^{(i)} \mid i = 1, \ldots, n,\ j = 1, 2,\ k = 1, \ldots, 2n-2\} \cup \\ \cup \{b_1, \ldots, b_{2n-2}\} \cup \{c\}.$$

Let H_0 be the complete graph whose vertex set is $\{x_1, \ldots, x_n, a_1, \ldots, a_{n-1}\}$ and let $H^{(i)}$ be the complete bipartite graph with partite sets $\{y_{1,1}^{(i)}, \ldots, y_{1,2n-2}^{(i)}\}$ and $\{y_{2,1}^{(i)}, \ldots, \ldots, y_{2,2n-2}^{(i)}\}$. Then $E(G(n))$ is

$$E(G(n)) = E(H_0) \cup \left(\bigcup_{i=1}^{n} E(H^{(i)}) \right) \cup \{x_p y_{1,k}^{(i)} \mid 1 \leqq i,\ p \leqq n,\ p \neq i,\ 1 \leqq k \leqq 2n-2\} \cup \\ \cup \{a_q y_{1,k}^{(i)} \mid 1 \leqq q \leqq n-1,\ 1 \leqq i \leqq n,\ 1 \leqq k \leqq 2n-2\} \cup \\ \cup \{ca_q \mid 1 \leqq q \leqq n-1\} \cup \{b_l y_{2,k}^{(i)} \mid 1 \leqq l \leqq 2n-2,\ 1 \leqq i \leqq n,\ 1 \leqq k \leqq 2n-2\} \cup \\ \cup \{c y_{1,k}^{(i)} \mid 1 \leqq k \leqq 2n-2,\ 1 \leqq i \leqq n\}.$$

Then $G(n)$ is $(2n-2)$-geodetically connected. Let $y_i = y_{2,1}^{(i)}$. Then $d_{G(n)}(x_i, y_i) = 3$. On the other hand, it is easy to see that every $x_i y_i$ geodesic passes through some a_j. Hence $G(n)$ has no desired geodetic linkage. ∎

For positive integer n, we define $h(n)$ by
$h(n) = \min \{k \mid$ every k-geodetically connected graph is geodetically n-linked$\}$. Then Corollary, Theorems 3 and Theorem 4 say that $h(n) = 3n-2$.

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